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A methodology to estimate the interest rates yield curve in Illiquid Market: the Tunisian case

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Abstract

The aim of this paper is to develop a methodology to estimate the interest rates yield curve and its dynamics in the Tunisian bond market, which is considered as an illiquid market with a low trading volume. To achieve this, first, we apply the cubic spline interpolation method to deal with the missing observation problem. Second, we focus on the work of Vasicek (1977) and Cox-Ingersoll and Ross (CIR) (1985), we estimate each model and discuss its performance in predicting the dynamics of the interest rates yield curve using Ordinary Least Squares and Maximum Likelihood Estimation (OLS and MLE) methods. The data sample consists of Treasury bond prices for the period from 14 July 2004 to 10 September 2012 collected from over the counter market. This assessment is done by Matlab software using the specified algorithms. The results suggest that the cubic spline method is accurate for estimation the average yield curve of interest rates. Then, the estimation of Vasicek (1977) and CIR (1985) models generate an upward sloping yield curves and seems suitable to replicate the stylized facts of the interest rates in the Tunisian bond market. Finally, the forecasting of the yield curve predicts an economic growth in the future characterized by a higher inflation

Keywords: Illiquid Market, Yield Curve of Interest Rates, Cubic Spline, Vasicek (1977) model, CIR (1985) model, OLS, MLE

1. Introduction

This paper is an attempt to develop a methodology to estimate a yield curve of the interest rates and its dynamics in the Tunisian bond market. During the last decade, the efficiency of the financial market has been improved through various reforms. The objective of the Tunisian authorities is to promote the development of the bond market by generating new products such as the long term Treasury bonds (BTA), the short term Treasury bonds (BTC) and zero coupon Treasury bonds, as well as the establishment of a group of primary dealers (SVT) responsible for the animating of the market. More than 90% of the issued bonds are government securities and the number of corporate issuers is still limited although fiscal incentives have been offered to induce listing. The bonds are usually listed on the Stock Exchange and therefore regulated and supervised by the Financial Markets Council (CMF). However, the bond market was marked by a significantly lower volume of trading, about 1,678 MTD on the Stock Exchange official quotation. A decline by 37.9% compared to 2010 is affected particularly by the suspension of trading on two occasions over the first quarter of 2011 and shorter trading sessions at the Stock Exchange. Therefore, the Tunisian bond market is smaller and illiquid compared to other bond markets of the developing countries where only about a handful of liquid bonds gets traded a day. In fact, the sparseness or infrequency[1]of daily Treasury bonds transactions explains in consequence the inaccuracy of the interest rates yield curve on the bond market. The drawback is that there is no specific term structure model of interest rate and the market operators devise a proxy of the yield curve based only on the liquid bonds. Thus, the unevenly distributed maturities of the different bonds make the estimation very difficult and the market less likely to form an entire and smooth yield curve.

However, most researches have been successfully applied to the developed bond markets, particularly the US bond market whereas little attention has been paid to the emerging markets (see for instance, Jian-Hsin Chou et al. (2009), Gonzalo Cortazar et al. (2007), Chi Xie et al. (2006), Dutta et al (2005), Alper et al (2004)). The existence of bond prices for a wide range of different maturities in the developed bond markets makes it easy to extract a yield curve of interest rates that explains observed prices. The developed bond market is generally well established and composed of relatively liquid securities with short and long maturities, in some countries such as the United States, zero-coupon bonds (Strips) of different maturities are individually traded.

The purpose of this paper is twofold: first, we estimate the interest rates yield curve using the Tunisian Treasury bond price data. We apply the cubic spline method to address the problem of missing observations in order to construct a smooth and continuous yield curve. The choice of this approach can be explained by its flexibility and easiness to implement as opposed to the other statistical methods. In addition, it can be used both for interpolation and smoothing. Second, we concentrate on the work of Vasicek (1977) and CIR (1985). We calibrate each model and discuss its performance in predicting the yield curve of the interest rates using the OLS and MLE methods. Therefore, one of the key points in this paper is to compare these models in terms of their ability to find a smooth yield curve which replicates the stylized facts of the interest rates on the Tunisian bond market.

This paper is organized as follows. The next section is devoted to a literature survey. Section 3 describes the implementation of the cubic spline method. Section 4 presents the estimation of the Vasicek and CIR term structure models. The empirical results are discussed in section 5. Section 6 concludes the paper.

2. Literature Review

The yield curve or the term structure of the interest rates describes the relationship between the yield of a bond and its maturity. It is the most important concept in pricing all the fixed income securities and interest rate derivatives such as bond options, caps, swaps etc, whose payoffs are strongly dependent on the interest rates. In addition, the yield curve reflects the monetary stance and inflation expectation. It is also a key indicator monitored by financial institutions to manage the interest rate risk and make investment decisions. Therefore, the research of how to estimate a fitted yield curve has become a very important issue and captures both the academic and practical

[1] Trading frequency is defined as the number of days for which we have at least one transaction of a Treasury bond of a specific maturity over all available trading days.

interests. Practitioners prefer an approach that is accessible, straightforward to implement and as accurate as possible. In general there are two distinct approaches to estimate the yield curve of the interest rates. The first approach includes the models of fitting the yield curves to the market data using statistical methods. The main purpose of these models is to find a smooth function between yields of bond prices and time to maturity. One of the yield curve estimation methods is the Bootstrap initiated by Bliss and Fama (1987) from discrete spot rates to fit a smooth and continuous yield curve to the market data. However, various curve fitting spline methods have been introduced. The most popular example of these procedures is the seminal work of MacCulloch (1971, 1975) which focused on estimating zero-coupon yields and discount factors using the polynomial splines. He found that the discount function could be fitted very well by cubic or higher order splines and the estimated forward rates are a smooth function. Vasicek and Fong (1982) try to use a third order exponential spline to calibrate the discount function and show that these models have a better fitting performance than the polynomial splines models. Then, Nelson and Siegel (1987) and Svensson (1994, 1996) suggested parametric curves that are flexible enough to describe a whole family of the observed yield curve shapes. However, this approach takes a static view targeting solely the shape of the yield curve of the interest rates.

In the second approach, the time dimension is concerned. An enormous literature focused on constructing and estimating the term structure models by specifying particular functional forms for the dynamics of the interest rates. In fact, these models postulate explicit assumptions about the evolution of the factors driving interest rates and deduce characterizations of shapes and movements of the yield curve in frictionless markets having no arbitrage. One-factor models are the first step in modelling the term structure of interest rates. These models are grounded on the estimation of bond yields as functions of the short term interest rates. Vasicek (1977) is one of the first term structure models which was used extensively in valuing bond options, futures and other fixed income derivatives. It remains the benchmark core model for its simplicity in estimation and possibility to easily calculate bond prices and yields. Thereafter, a famous extension of the Vasicek model is the CIR (1985) model, which uses an intertemporal general equilibrium asset pricing model to study the term structure of the interest rates. The model provides solutions for bond prices and a complete characterization of the term structure which incorporates risk premiums and expectations for future interest rates. Subsequently, the model is extended to include multiple factors and to value nominal bonds and nominal claims. Longstaff and Schwartz (1992) develop a two factor model using the interest rate and its instantaneous variance as state variables to derive a closed form of expressions for the price of the discount bonds and discount bond options.

The major benefit of these models is to provide a link between intertemporal asset pricing theory and the term structure of the interest rates that produces a frequently convenient closed form of solution for asset prices. However, all these models generally imply a term structure of the interest rates conflicting with the market yield curve. In fact, the drift and volatility of the interest rates and the market price of risk are considered as the single source of uncertainty to determine the dynamics of the term structure. This problem is solved by Ho and Lee (1986) and Hull and White (1990) who used different information to characterize the yield curve dynamics. The information set includes the spot interest rate, volatility and the functional form of the yield curve. The model of Heath, Jarrow and Morton (1992) also allows capturing the full dynamics of the entire yield curve in an arbitrage free framework. This is an extension of one factor model developed by Ho and Lee (1986) at a multi factor model by considering forward rates rather than bond prices.

3. The implementation of cubic spline method

3.1 Data construction and analysis

The data[2] are used on a daily basis. They, generated used the pricing of 31Tunisian treasury bonds[3] (10 long term Treasury bonds 'BTA' and 21 short term Treasury bonds 'BTC'). The short term maturity treasury bonds are the government treasury instruments for the maturities of less than one year as issued with 13, 26 and 52 weeks maturity. Currently, only the BTC of 52 weeks are listed on the market. However, the BTA are the long term public debt support. Their maturity varies between 6, 7, 10, 12 and 15 years.

The data run from 14 July 2004 to10 September 2012 thus in total 1050 observations. The yield to maturity[4] for all the Treasury bonds is not directly observable on the market, which means that they need to be computed[5] from the market prices of the Treasury bonds using the formula of excel spreadsheet.

Empirical analysis covers the period from 12 April 2010 to 10 September 2012 providing 137 observations in total. It is a short sample due to the sparse Treasury bond prices. Then, the data describe the historical changes of the redemption yield of a Treasury bond with maturities: 3 months, 6 months, 9 months, 1...14-year. However, the 52-week Treasury bond yields are considered a better approximation of the short interest rate. A lot of studies on the yield curve of the interest rates were conducted in the developed countries like the USA, Japan, Germany and others. In these studies, the three-month Treasury bonds are considered a better approximation of the short interest rate. For instance, Bernnan and Schwartz (1982) applied a model for the pricing of the US Treasury bonds from 1948 to 1979. Chan Karolyi, Longstaff and Sanders (1992) compared different models using the Treasury bill yields as a proxy to select the best fitting model of the short term interest rates. Mato (2005) analyzed the immunization portfolio against the interest rate risk from the mid 1990 and early 2000 on US Treasury bond market. Munnick and Schotman (1994) also developed a model for the short term interest rate on the Dutch bond market.

Figure 1 illustrates the evolution of the 52-week Treasury bond yields which are used to estimate and simulate the short interest rate processes.

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[2] The database is collected from International Arab Tunisian Bank (BIAT).

[3] These bonds are issued by auction on the capital market for a 1.000 dinars value and negotiable at all banks intervening on the money market.

[4] The day count basis convention used for BTA is actual / 365 and actual/360 for BTC.

[5] To understand the yield to maturity, one must have that the market price of a bond is equal to the present value of its future cash flows, as showing in the following formula: $P=\sum_{t=1}^{n}\frac{c}{(1+r)^t}+\frac{F}{(1+r)^n}$ Where; P : Market price of the Treasury bond, n : Number of periods, c : Coupon payment, r : yield to maturity, F : Maturity value, and t : Time period when payment is to be received

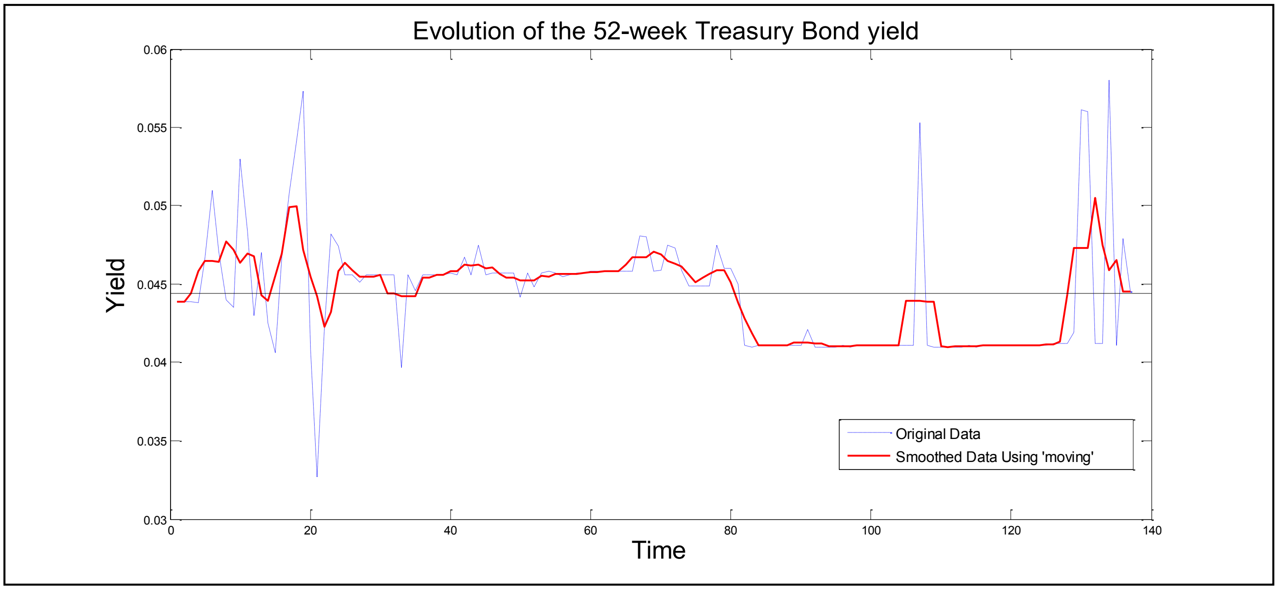

Fig.1: Evolution of the 52-week Treasury bond yields

The descriptive statistics show that the Jarque-bera test for normality rejects the null hypothesis with p-value 1e-003 (0.1%). Indeed, the data of 52-week Treasury bond yields have respectively coefficients of skewness and kurtosis of 1.1312 and 5.7325 which are different from a normal distribution. Moreover, the Augmented Dukey-Fuller (ADF) test applied to the data rejects the null hypothesis at the 1% risk level, which means that this series is stationary. The results are reported in table 1.

Table1: Result of ADF test for 52-week Treasury bond yields

Variable	**ADF statistic**	**Critical value**		
		1%	5%	10%
52-weeks Treasury bond yields	-7.306436***	-3.481623	-2.883930	-2.578788

Symbol (***) denotes 1% level of significance.

The goodness of fit of the data is measured by the moving average filter. A moving average smooths out the random fluctuations in the data to show a pattern or trend more clearly. Thus, the average values from a specific number of data points (set by the period option) are used as a point in the trendline. If the period is set to 2, for instance, the average of the first two data points is used as the first point in the moving average trendline. The average of the second and third data points is used as the second point in the trendline, and so on.

3.2 Cubic spline method

In mathematics, a cubic spline is a piecewise polynomial function of order three which is twice differentiable at each point known as knot points or nodes. To construct a set of cubic splines, let the function $R_i(t)$ denote the cubic polynomial associated with the t segment $[t_i, t_{i+1}]$; $R_i(t) = a_i(t-t_i)^3 + b_i(t-t_i)^2 + c_i(t-t_i) + r_i$. Where $i = 1,2...n$ is the number of market observations, r_i represents knot point i, and t_i represents the time to maturity of market observation i, $n-1$ splines, and three coefficients per spline. The coefficients of the cubic spline function defined over the interval $[t_i, t_{i+1}]$ can be obtained by imposing the following constraints:

$$a_i(t_{i+1}-t_i)^3 + b_i(t_{i+1}-t_i)^2 + c_i(t_{i+1}-t_i) = r_{i+1} - r_i$$

$$3a_{i-1}(t_i - t_{i-1})^2 + 2b_{i-1}(t_i - t_{i-1}) + c_{i-1} - c_i = 0$$

$$6a_{i-1}(t_i - t_{i-1}) + 2b_{i-1} - 2b_i = 0$$

$$b_1 = 0$$

$$6a_{n-1}(t_n - t_{n-1}) + 2b_{n-1} = 0$$

The first set of $n-1$ constraints requires that the spline function join the knot points perfectly. The second and third set of $2n-2$ constraints require that the first and second derivative constraints match the adjacent splines. Finally, the last two constraints are end point constraints that set the derivative equal to zero at both ends. Thus, the linear algebraic system consists of $3n-3$ equations and $3n-3$ unknowns coefficients can be solved to produce an optimal piecewise cubic spline. Therefore, the use of a piecewise cubic spline technique that passes through the observed market data points creates a fitted smooth yield curve and avoids kinks. MacCulloch (1971) introduced the spline method to accommodate the data structure by dividing the maturity range into subintervals so that the number of the Treasury bonds in each subinterval is roughly the same and the spline functions are connected to ensure continuity and differentiability at the nodes.

However, the linear interpolation is simple to implement and closely tracks observed the market interest rates. All the observed market data points are connected by a straight line to form a complete yield curve. As a consequence, linear interpolation can be presented in a closed form, which simplifies the interpolation process. $R(t) = R(t_i) + \left[\frac{(t-t_i)}{(t_{i+1}-t_i)}\right] \times \left[R(t_{i+1}) - R(t_i)\right]$. Here, i is the market observation index with time to maturity of t_i, and $R(t)$ represents the yield corresponding to maturity t, where $t_i \le t \le t_{i+1}$. Linear interpolation is inappropriate for modeling the yield curves that change the slope frequently and tend to produce kinks around transition areas. In fact, the point between two nodes cannot be accurately estimated using a straight line.

Moreover, Treasury bonds trade infrequently so that for every particular day there are Treasury bond prices for only a few maturities. This missing observations problem makes it difficult, and sometimes impossible, to estimate the yield curve using only current data. For this reason we determine the average yields of the Treasury bonds for each maturity in order to construct a continuous curve joining market observed data as smoothly as possible.

We try to apply the linear and cubic spline interpolation methods to complete the missing data. Therefore, the performance of the cubic spline method in estimating a smooth curve is better than the linear method, as depicted in figure 2. Indeed, the estimation of data using cubic spline has the advantage that a large number of data points can be connected. This is the most straightforward means that gives meaningful data to construct a correct interest rate yield curve in the Tunisian bond market.

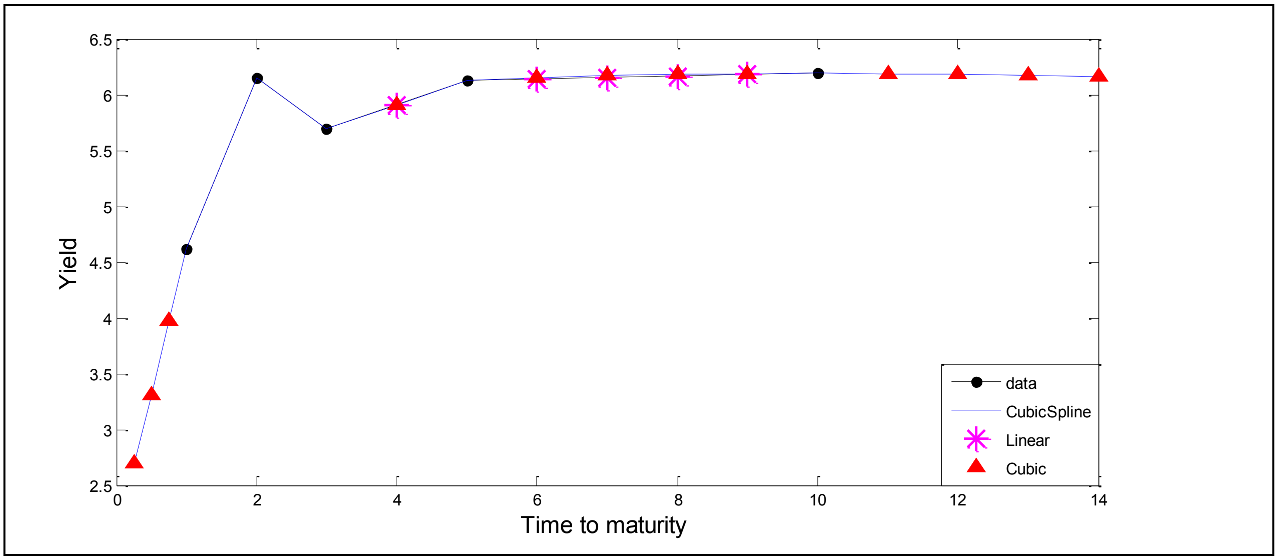

Fig.2: Interpolation methods to plot yield curve

Figure 3 plots the average yield curves for nine years. We can also infer that the cubic spline interpolation seems able of generating a general trend of interest rates on the Tunisian bond market. All in all, we obtain upward shaped yield curves with oscillations which is different from the observed average yield curve[6]. The CMF has undertaken, with the market participants, a simple reflection to construct a yield curve. This curve based on the auction price of BTA and BTC is arbitrary and imprecise.

The average yield curves from 2004 to 2010 are all an upward trend and the position of the curves changes in the whole sample period. The plots of the dots show low differences. Nevertheless, the yield curve of year 2011 revealed that the changes of the yields of all the maturities follow first a flat and alter a downward trend. The downward sloped is due to the dots estimated which is often a harbinger of an economic slowdown. Therefore, the shape of the splined curve affects its smoothness.

[6] The yield curve is diffused via the Tunisian Depository Learning (STICODEVAM) and the Financial Market Council (CMF).

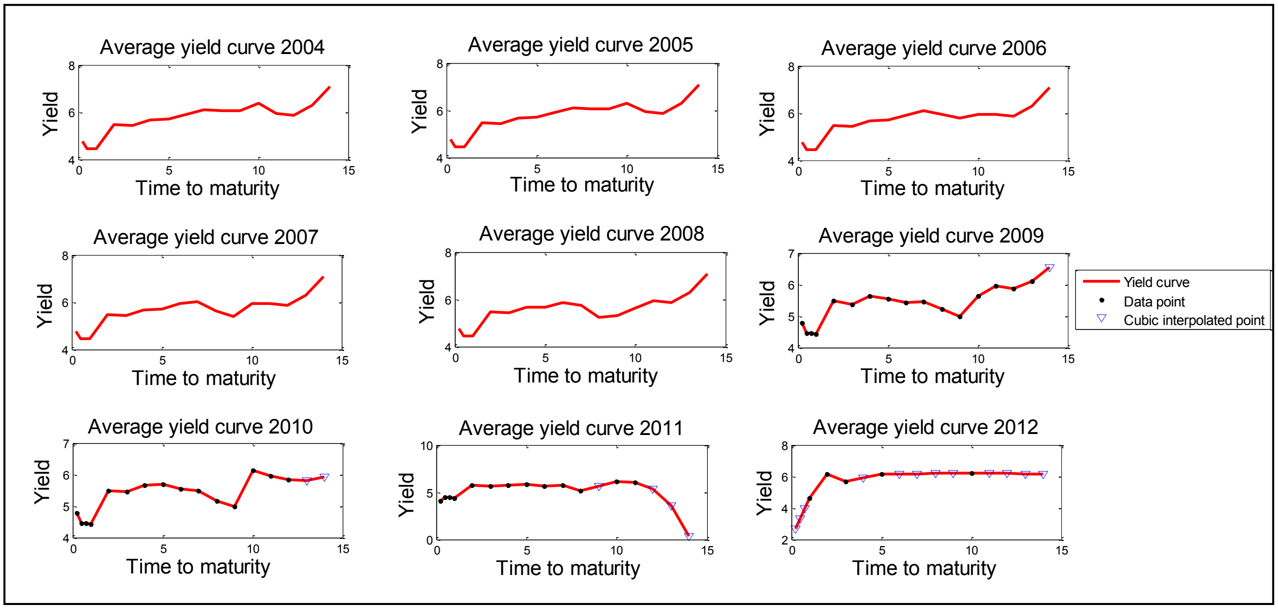

Fig.3: Plots of the average yield curves of interest rates

Moreover, when estimating the curve by an interpolation between the nodes, the user must consider the conflicting issues. There is a need to balance between simplicity and correctness, and hence a tradeoff between the ease of the use and the accuracy of the result. In this case, we can see that the cubic spline does not result in a smooth curve. We accept a lower degree of accuracy at the nodes, in favour of smoothness across the curve. Therefore, we recommend using a cubic spline method which ensures that the curve passes through all the discrete data points. This enables practitioners to fit a reasonable and accurate yield curve to the observed market interest rates, but it may oscillate wildly. However, we need to use parametric models in order to estimate a continuous yield curve from a set of discrete observations and eliminate the oscillations.

4. Estimation of the Vasicek and CIR models

The Vasicek (1977) and CIR (1985) models assume that the interest rate is a diffusion process, defined by the following stochastic differential equations:

$$dr_t = \theta(\mu - r_t)dt + \sigma dW_t, \qquad (1)$$

$$dr_t = \theta(\mu - r_t)dt + \sigma\sqrt{r_t}\,dW_t, \qquad (2)$$

Where $W(t)$ is a wiener process which models the random market risk factor[7] and parameter σ determines the volatility of the interest rate. Both models specify that the interest rate tends to revert to its long term mean μ, and θ can then be interpreted as the speed of this reversion. One

[7] We assume that the market price of risk $\lambda = 0$.

of the drawbacks of the Vasicek model is that it can admit negative values of the interest rate. The CIR model resolves this unfortunate feature by introducing the square root term $\sqrt{r_t}$.

The Vasicek yield curve of the interest rates in the risk neutral world is extracted from the following equation:

$$R(t,T) = \left(\mu - \frac{\sigma^2}{2\theta^2}\right) + \left(r_t - \mu + \frac{\sigma^2}{\theta^2}\right)\frac{1-e^{-\theta(T-t)}}{\theta(T-t)} + \frac{\sigma^2}{4\theta^3(T-t)}\left(1-e^{-\theta(T-t)}\right)^2$$

(3)

The yield curve of the interest rates in the risk neutral world deriving from the CIR model is equal to:

$$R(t,T) = \frac{1}{T-t}\left[\left(B(t,T).r_t\right) - \log\left(A(t,T)\right)\right]$$

(4)

Where; $$A(t,T) = \left[\frac{2\gamma \exp\left\{\frac{1}{2}(\theta+\gamma)(T-t)\right\}}{2\gamma + (\theta+\gamma)\left(\exp\{\gamma(T-t)\}-1\right)}\right]^{2\theta\mu/\sigma^2}$$

$$B(t,T) = \frac{2.\exp\left(\{\gamma(T-t)\}-1\right)}{2\gamma + (\theta+\gamma)\left(\exp\{(T-t)\gamma\}-1\right)}$$

And $$\gamma = \sqrt{\theta^2 + 2\sigma^2}$$

A common practice is to calibrate and predict the interest rates term structure models. Several advanced econometric methods have been developed. For example, the results of Brown and Dybvig (1986) in examining the CIR (1985) model with cross sectional data are declined by Gibbons and Ramswamy (1993) using the generalized method of moments (GMM) on short term Treasury bills data, but they are accepted by Pearson and Sun (1994) using the MLE. Chan, Karolyi, Longstaff and Sandres (1992) used the GMM method to estimate and compare a variety of term structure models on the US market. Their results demonstrate that Vasicek (1977) and CIR (1985) models perform poorly compared to Dothan (1978) and CIR (1980) models. Jensen (2000) has demonstrated, through a new implementation of the Efficient method of moment (EMM) estimation principle, that the Longstaff and Schwartz (1992) model is inadequate for the description of the interest rates. The EMM proposed by Gallant and Tauchen (1996) is the best alternative widely used in the several well known models when the MLE method is impossible, for example, the stochastic volatility models (Andersen and Lund (1997)), affine term structure models (Dai and Singleton, 2000) and quadratic term structure models (Ahn et al., 2002). However, the non-parametric procedure is proposed by Ait Sahalia (1996) to estimate the diffusion coefficient using 7 days Eurodollar deposit rates from the Bank of America during the period from June 1973 to February 1995. The main results are that the fit is imperfect and the tests reject every parametric model of the spot rate. Stanton (1997) finds nonlinearities in the drift coefficient of one factor models using daily data rate of 3 months of the U.S. Treasury bills from 1965 to 1995.Other approaches have been applied to the empirical estimation of interest rate term structure models such as Kalman filter and Markov Chain Monte Carlo (MCMC) methods. In recent years, Bayesian estimation has also paid attention to the development of MCMC

simulation for calibrating the interest rate models. This Bayesian approach was inflexible in the past decade due to the difficulty of its implementation for large scale problems.
My assignment is to estimate the parameters of Vasicek (1977) and CIR (1985) interest term structure models in risk neutral world using the OLS and MLE techniques.

4.1 Ordinary least squares

Vasicek (1977) model is equivalent to a first order autoregressive AR (1) model. The discretized version with time step $\Delta t = t_i - t_{i-1}$ is: $r_{t_i} = c + b r_{t_{i-1}} + \delta \xi_{t_i}$ (5)

Where the coefficients are; $c = \mu\left(1 - e^{-\theta \Delta t}\right)$

$$b = e^{-\theta \Delta t}$$

$$\delta = \sigma \sqrt{\left(1 - e^{-2\theta \Delta t}\right) / 2\theta}$$

And ξ_t is a Gaussian white noise. Indeed, the following expressions for θ, μ and σ hold,

$$\hat{\theta} = \frac{-\ln(b)}{\Delta t} \quad (6)$$

$$\hat{\mu} = \frac{c}{1-b} \quad (7)$$

$$\hat{\sigma} = \delta / \sqrt{\left(b^2 - 1\right) \Delta t / 2\ln(b)} \quad (8)$$

Similarly for the CIR model described by equation (2), the discretised equation is given by:

$$r_{t+\Delta t} - r_t = \theta(\mu - r_t)\Delta t + \sigma \sqrt{r_t}\, \xi_t , \quad (9)$$

with ξ_t is a Gaussian white noise process. For performing OLS we transform (9) into: $\frac{r_{t+\Delta t} - r_t}{\sqrt{r_t}} = \frac{\theta \mu \Delta t}{\sqrt{r_t}} - \theta \sqrt{r_t} \Delta t + \sigma \xi_t$. Then, the drift initial estimates are found by minimizing the OLS objective function:

$$\left(\hat{\theta}, \hat{\mu}\right) = \underset{\theta, \mu}{\arg\min} \sum_{i=1}^{N-1} \left(\frac{r_{t_{i+1}} - r_{t_i}}{\sqrt{r_{t_i}}} - \frac{\theta \mu \Delta t}{\sqrt{r_{t_i}}} + \theta \sqrt{r_{t_i}} \Delta t \right)^2 \quad (10)$$

The diffusion parameter initial estimate $\hat{\sigma}$ is found as a standard deviation of residuals. Thus, the MATLAB code implementation can solve the problem quite easily. Since our data set consists of daily observation, so Δt can be taken as 1/250.

4.2 Maximum likelihood

One of the alternative methods is to estimate the parameters which maximise the likelihood function.

Given N observations of 52-week Treasury bond yields $\{r_{t_i}, i = 1....N\}$. The likelihood function is as follow: $L(\psi) = \prod_{i=1}^{N-1} p(r_{t_{i+1}} \mid \mathrm{r}_{\mathrm{t_i}}; \psi; \Delta t)$

(11)

with Δt time step; $\psi \equiv (\theta, \mu, \sigma)$ a parameter vector to be estimated and $p(r_i \mid \psi)$ defined as the transition function of the Vasicek and CIR process respectively. Then, the log-likelihood function is, $\ln L(\psi) = \sum_{i=1}^{N-1} \ln p(r_{t_{i+1}} \mid \mathrm{r}_{\mathrm{t_i}}; \psi; \Delta t)$

(12)

Therefore, the maximum likelihood estimator $\hat{\psi}$ of parameter vector ψ is: $\hat{\psi} \equiv (\hat{\theta}, \hat{\mu}, \hat{\sigma}) = \arg\max_{\psi} \ln L(\psi)$

(13)

Moreover, the application of the maximum likelihood requires the specification of the transition function of each process. Hence, the conditional density function for Vasicek model is given by:

$$p(r_{t+\Delta t} \mid r_t; \psi, \Delta t) = \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left[-\frac{\left(r_{t_{i+1}} - r_{t_i} e^{-\theta\Delta t} - \mu\left(1 - e^{-\theta\Delta t}\right)\right)^2}{2\hat{\sigma}^2} \right]$$

(14)

With, $\hat{\sigma}^2 = \sigma^2 \frac{1 - e^{-2\theta\Delta t}}{2\theta}$.

The corresponding log-likelihood function is:

$$\ln L(\psi) = -\frac{N-1}{2}\ln(2\pi) - \frac{N-1}{2}\ln(\hat{\sigma}) - \frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{N-1}\left[r_{t_{i+1}} - r_{t_i} e^{-\theta\Delta t} - \mu\left(1 - e^{-\theta\Delta t}\right)\right]^2$$

(15)

Now, let us focus on the MLE of the CIR model. The transition density function for the CIR process has a closed form expression.

$$p(r_{t+\Delta t} \mid r_t; \psi, \Delta t) = c e^{-uv} \left(\frac{u}{v}\right)^{\frac{q}{2}} Iq\left(2\sqrt{uv}\right),$$

(16)

Where;

$$c = \frac{2\theta}{\sigma^2\left(1 - e^{-\theta\Delta t}\right)},$$

$$u = c r_t e^{-\theta\Delta t},$$

$$v = c r_{t+\Delta t},$$

$$q = \frac{2\theta\mu}{\sigma^2} - 1,$$

And $I_q\left(2\sqrt{uv}\right)$ is modified Bessel function of the first kind of order q. Feller (1951) shows that the distribution function of the interest rates is a non central chi-squared with $2q + 2$ degrees of freedom and non central parameter $2u$. Hence, the log-likelihood function can be derived from the conditional density function as:

$$\ln L(\psi) = (N-1)\ln c + \sum_{i=1}^{N-1}\left\{-u - v + 0.5q\ln\left(\frac{v}{u}\right) + \ln\left\{Iq\left(2\sqrt{uv}\right)\right\}\right\} \quad (17)$$

Therefore, a numerical solution based on the function fminsearch[8] in MATLAB is executed to solve the optimisation problem (13) and (17).

5. Inference - Estimation result analysis

The estimation of parameter vector $\psi \equiv (\theta, \mu, \sigma)$ is carried out using the OLS regression and MLE techniques for the 52-week Treasury bond yields. Table 2 shows the parameter estimates of vasicek (1977) and CIR models, respectively.

Table 2: Parameter estimation of Vasicek and CIR models

Model	Vasicek			CIR		
Parameter	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$
OLS	0.0037	0.0444	3.2872e-004	0.0023	0.0444	0.001
MLE	0.0037	0.0444	3.2629e-004	0.0037	0.0444	0.0015
Jackknife	0.0037	0.0444	3.2639e-004	0.0037	0.0444	0.0015

The OLS and the MLE produce the same values of parameter estimates with low standard deviation. In fact, the most relevant problem in the present context is the estimation bias in the practical use of econometric estimators for the continuous time discretised models. In general, discretisation introduces an estimation bias since the internal dynamics between the sampling points are ignored. Thus, the misspecification bias results in an inconsistent estimation of the parameters of the continuous time model with consequent bias effects on the shape of the yield curve. Like the OLS using a naive regression, the maximum likelihood estimator can have a substantial bias in the estimation of continuous time models, such as diffusion models of short the interest rates. For example, it is well known that the Black Sholes stock option price estimates are biased. The bias is mainly found in the drift parameter which happens when the mean reversion parameter is small. As mentioned by Yu and Phillips (2005), the MLE and the OLS produce a bias of more than 200% in the bond price. In spite of its generally good asymptotic properties, the approximation of the maximum likelihood estimator is generally inconsistent. Consequently, it is extremely important to estimate these parameters without any bias. Furthermore, Ait Sahalia (2008) and Tang and Chen (2009) used bootstrapping to reduce MLE bias. Bekaert, Hodrick, and Marshall (1997) used Kendall's method to correct for bias in testing the expectations hypothesis of the interest rates yield curve. In this context, the jackknife technique is applied to assess the variability of the MLE for the Vasicek and CIR parameter estimates.

The bias of MLE is determined as: $bias_{jack} = (n-1)\left(\hat{\theta}_{(.)} - \hat{\theta}\right)$. Where: $\hat{\theta} = S(x)$ an estimator for data set $X = (x_1, x_2, ..., x_n)$; $\hat{\theta}_{(i)} = S(x_i)$ for $i = 1,2,...,n$ be the i^{th} jackknife replication of $\hat{\theta}$ and

[8] This function is based on the Nelder-Mead simplex method.

$\hat{\theta}_{(.)} = \frac{\sum_{i=1}^{n} \hat{\theta}_{(i)}}{n}$. The jackknife estimate of standard error will then be defined by: $\hat{se}_{jack} = \left[\frac{n-1}{n} \sum \left(\hat{\theta}_{(i)} - \hat{\theta}_{(.)} \right)^2 \right]^{1/2}$

Table 3: Jackknife bias corrected of estimated parameters

Model	Vasicek			CIR		
Parameter	$Bias_{jack}$	Se_{jack}	$\hat{e}_{jack}$	$Bias_{jack}$	Se_{jack}	$\hat{e}_{jack}$
$\hat{\theta}$	0	0	0.0037	0	0	0.0037
$\hat{\mu}$	-1.4155e-014	1.2182e-015	0.044	-1.4155e-014	1.218e-015	0.0444
$\hat{\sigma}$	-1.0461e-007	9.0032e-009	3.2639e-004	-1.0461e-007	9.0032e-009	0.0015

Table 3 shows that the standard error of the parameters is very small, so the maximum likelihood estimator is unbiased and the parameters of the model are estimated very accurately. On the other hand, the jackknife produces the same values of the MLE parameter estimates. Conversely, the standard deviation of the sample remains low. Indeed, our findings indicate that the jackknife allows bias reduction in the parameter estimates under the model misspecification without increasing the variance.

However, if we plot a histogram for each estimation method based on simulation, no preference between OLS and jackknife is observed (figure 4 and 5).

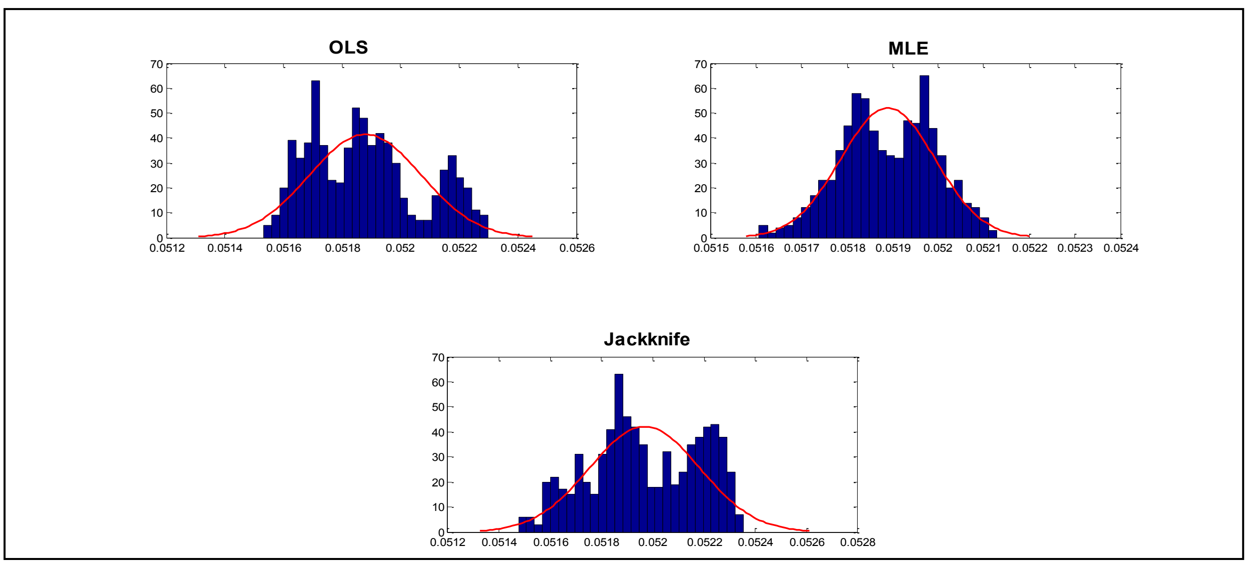

Fig.4: Distribution fit of three methods of the Vasicek estimation model based on simulation.

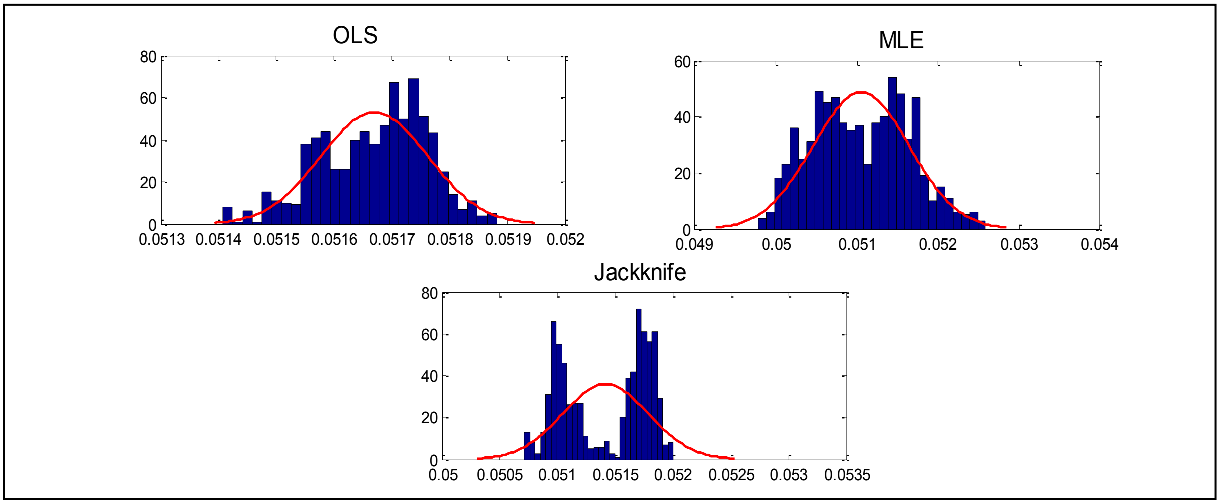

Fig.5: Distribution fit of three methods of the CIR estimation model based on simulation.

As a result, the MLE has little asymmetry than the OLS, so it's the best method of estimating the term structure model parameters.

Figure 6 shows a steep upward sloping yield curve, which means that the Treasury bond yields rise as maturity extends.

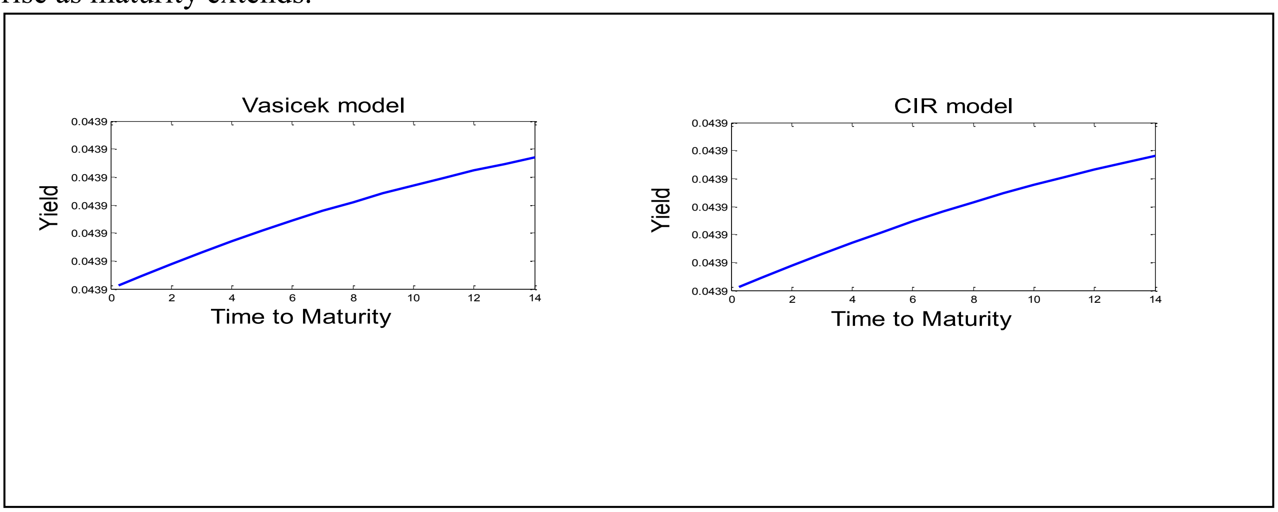

Fig.6: Estimates of Vasicek and CIR yield curves

Indeed, the resulting yield curve of CIR (1985) model is very close to that of Vasicek (1977). A sharply upward sloping has often preceded an economic upturn and the interest rates will rise significantly in the future. Under this situation, if investors hold investment, they may receive a higher rate in the future. However, the empirical findings specified by these models are in line with the stylized fact of the interest rate yield curve estimated using the cubic spline method.

6. Forecasting analysis

The simulation of the term structure models is based on the discretisation scheme as described above. In fact, we use a forecasting period of thirty years with a time step of one year. We run the simulation using the maximum likelihood parameter estimates for Vasicek and CIR models respectively. We start the simulation from the same point ($r_0 = 4.39\%$).

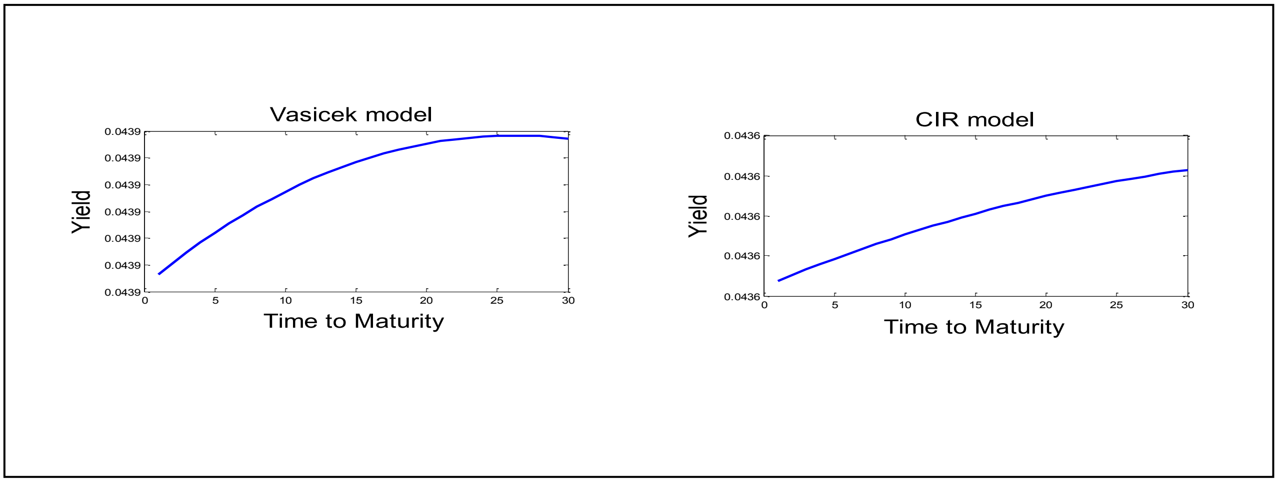

Fig.7: Simulation of Vasicek and CIR yield curves

The simulation of the yield curve with Vasicek and CIR models generates an upward sloping curve (Figure 7). This explains the stylized fact that the short term yields are lower than the long term yields, as shown in figure 8.

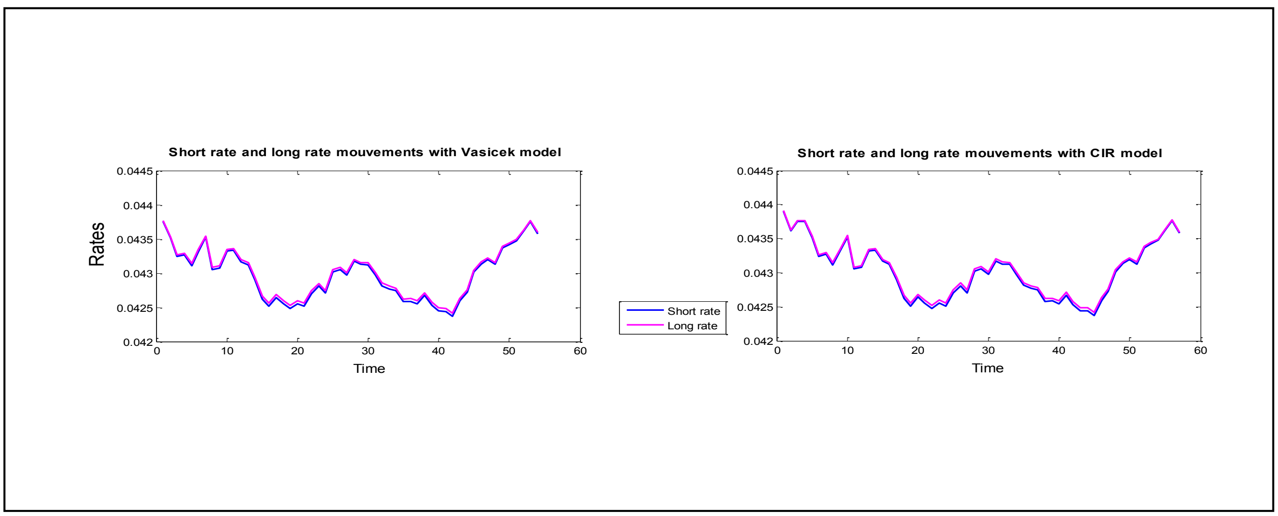

Fig.8: Spread between short and long term interest rates

The above figure compares the simulated long and short term Treasury yields. The spread is relatively small which results in yield curve steep. In fact, with Vasicek and CIR models the both rates move up and down somewhat together over time. Therefore, the parallel shifts are common, in line with the movements of the monetary market rate (TMM). The Central Bank of Tunisia (BCT) decided to reduce the money market rate to 3.76 percent in August 2011 which was still falling until 2012. Figure 9 illustrates the evolutions of the TMM rate from January 2010 to December 2012.

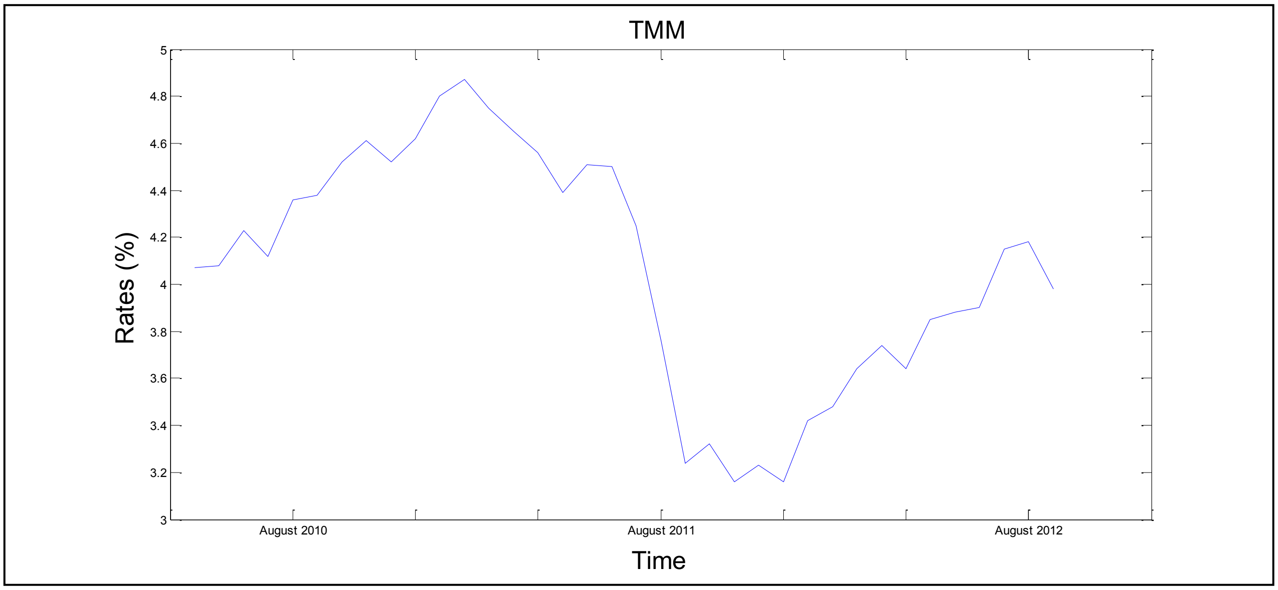

Fig.10: Evolution of monetary market rate

However, the positive slope reflects investors' expectations for an economic expansion and higher interest rates in the future. This anticipation leads to expectations that the policy makers will tighten monetary policy by raising the short term interest rates in the future to fight inflation. It also creates a need for a risk premium associated with the uncertainty about the future rate of inflation and the risk which poses for the future value of cash flows. The investors assess these risks into the yield curve by demanding higher yields for maturities further into the future. In consequence, the long term tendency reflects the expected future of short term yields over the holding horizon.
The simulated future short term interest rates based on the estimated Vasicek and CIR model parameters are exhibited in figure 11.

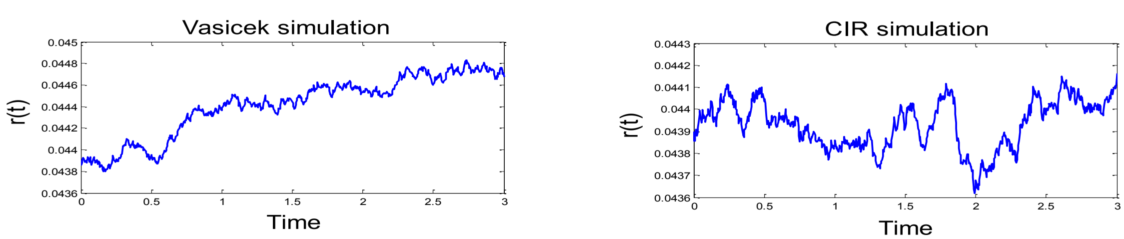

Fig.11: Simulation of the short term interest rates

A good model of the interest rates yield curve is vital for the smooth functioning of the market. Thus, the yield curves predicted by Vasicek (1977) or CIR (1985) model seem suitable to forecast the interest rate movements in the Tunisian market.

7. Conclusion

The Tunisian bond market is illiquid with a noticeably smaller trading volume compared to other bond markets of the developing countries. In this kind of market, the estimation of an accurate interest rates yield curve is very difficult. Therefore, this paper developed a methodology to estimate the interest rates yield curve and its dynamics in the Tunisian bond market. Our empirical analysis was carried out into two sequences: First, we constructed the yield curve of interest rates using a simple interpolation method. The yield curve is only known with certainty for a few specific maturity dates, while the other maturities are estimated using a cubic spline method. Second, we calibrated the parameters of Vasicek and CIR models through OLS and MLE methods using 52-week Treasury bond yields as a proxy of the short term interest rates. We tried to take into account the various particularities of the Tunisian bond market data.

The empirical results indicated that the cubic spline method is a tractable and reasonably correct estimation method that we recommend in any market with infrequent trading. We were able to construct an accurate average yield curve but not smooth. Then, the estimation of the Vasicek (1977) and CIR (1985) term structure models generate the same shape of the interest rates yield curve and show an ability to replicate the stylized facts of the interest rates in the Tunisian bond market. Finally, the simulation of these models predicts that the spread between the long term and the short term yields is very small which is often a harbinger of future economic growth with higher inflation.

This work is a good experience to estimate and predict the yield curve of interest rates in the Tunisian bond market where no yield curve exists. Therefore, this methodology can be used by the policy makers to draft its monetary policy and by a financial institution in its daily trading activities to predict possible losses associated with unfavourable movements in the interest rates.

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